XI TALLER DE CIENCIAS PLANETARIAS

Secular Resonance for Retrograde Orbits

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Secular Resonance: Asteroids experience secular resonances when the frequencies are resonant with one of the fundamental oscillation frequencies of the orbits of the planets that make up the system (similar to coupled harmonic oscillators).

Prograde Orbits: $0° < I < 90°$ **Retrograde Orbits**: $90° < I < 180°$
Lagrange's Planetary Equations [1]

Prograde

$$\dot{h} = +\frac{1}{na^2}\frac{\partial R}{\partial k}, \quad \dot{k} = -\frac{1}{na^2}\frac{\partial R}{\partial h}, \quad \dot{p} = +\frac{1}{na^2}\frac{\partial R}{\partial q}, \quad \dot{q} = -\frac{1}{na^2}\frac{\partial R}{\partial p}.$$

$$R = na^2\frac{1}{2}A(h^2+k^2)^2+\frac{1}{2}B(p^2+q^2)^2++\sum_{j=1} A_j(hh_j+kk_j)+\sum_{j=1} B_j(pp_j+qq_j)).$$

whose solution

$$h(t) = e_{free}\,\mathrm{sen}(At + \beta) - \sum_{i=1}^{2} \frac{\nu_i}{(A - g_i)}\,\mathrm{sen}(g_i t + \beta_i), \tag{1}$$

$$k(t) = e_{free}\cos(At + \beta) - \sum_{i=1}^{2} \frac{\nu_i}{(A - g_i)}\cos(g_i t + \beta_i), \tag{2}$$

$$p(t) = I_{free}\,\mathrm{sen}\,(Bt + \gamma) - \sum_{i=1}^{2} \frac{\mu_j}{(B - f_i)}\,\mathrm{sen}(f_i t + \gamma_i), \tag{3}$$

$$q(t) = I_{free}\cos\,(Bt + \gamma) - \sum_{i=1}^{2} \frac{\mu_j}{(B - f_i)}\cos(f_i t + \gamma_i). \tag{4}$$

Retrograde

$$\dot{h} = -\frac{1}{na^2}\frac{\partial R}{\partial k}, \qquad \dot{k} = +\frac{1}{na^2}\frac{\partial R}{\partial h}, \qquad \dot{p} = -\frac{1}{4na^2}\frac{\partial R}{\partial q}, \qquad \dot{q} = +\frac{1}{4na^2}\frac{\partial R}{\partial p}.$$

when $c = cos(I/2)$ and

$$R = na^2(\frac{1}{2}A(h^2+k^2)+\sum_{j=1}^{2}A_j(hh(j)+kk(j))+\frac{1}{2}B(p^2+q^2)-\sum_{j=1}^{2}B_j(pp(j)+qq(j))).$$

whose solution

$$h(t) = e_{free}\,\mathrm{sen}(-At+\beta) - \sum_{i=1}^{2}\frac{\nu_i}{(A+g_i)}\,\mathrm{sen}(g_i t+\beta_i), \tag{5}$$

$$k(t) = e_{free} cos(-At+\beta) - \sum \frac{\nu_i}{(A+g_i)}\cos(g_i t+\beta_i), \tag{6}$$

$$p(t) = c_{free}\,\mathrm{sen}(-Bt+\gamma) + \sum_{i=1}^{2}\frac{\mu_i}{B+f_i}\,\mathrm{sen}(f_i t+\gamma_i), \tag{7}$$

$$q(t) = c_{free}\cos(-Bt+\gamma) + \sum_{i=1}^{2}\frac{\mu_i}{B+f_i}\cos(f_i t+\gamma_i). \tag{8}$$

The frequency A is associated with the frequency of the e longitude of pericentre and the frequency B is associated with the frequency of the longitude of the ascending node, where $A = -B$.

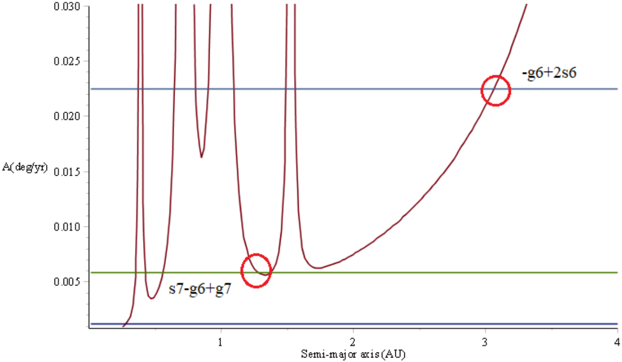

Figure 1: The variation of the frequency A as a function of semi-major axis, derived from perturbations by all planets. The frequencies $A + g_i = 0$ end $B + f_i = 0$ values are represented by horizontal lines.

Values of the studied frequencies. The frequencies values were obtained from the article. [2] The frequencies of the secular solution of the planets for h, k and p, q, respectively, are:

The frequencies associated with the longitude of pericentre are:

Frequency	Value (deg/yr)
$-g_6 + 2s_6$	-0,022476919
$g_5 - g_6 + g_7$	-0,005807473

The frequencies associated with the angle longitude of ascending node are:

Frequency	Value (deg/yr)
$s_7 - g_5 + g_6$	0,005833669
$-s_6 + 2g_6$	0,023004518

To confirm the theoretical locations of the secular resonances, we use n-body integrator MERCURY [3] to search for librations of the secular angles around the calculated value of the semimajor axis. The secular angles related to the apsidal precession and nodal precession are respectively expressed by $\varpi_a + g\,t$ and $\Omega_a - s\,t$, where in retrograde case $\varpi_a = \omega - \Omega$ and g , s are the frequencies of each case.

Changing the initial value of semimajor axis in small variations, we try to observe how the secular angles behave in the region. As the linear problem are considered, we use the following initial conditions to three different values of the argument of periapsis $\omega = 0°, 90°$ and $180°$.

$$e = 0.01\,,\ I = 179.999°\,,\ \Omega = M = 0°$$

Region around $a = 3.0$ au

Analysing the temporal variation of the secular angles with $g = -g_6 + 2\,s_6$ and $s = -s6 + 2\,g_6$, we can clearly see the dependence of eccentricity with the apsidal secular angle libration and the dependence of inclination with the libration of the secular angle related to nodal precession. The values of the analyzed frequencies are described in Table 1.

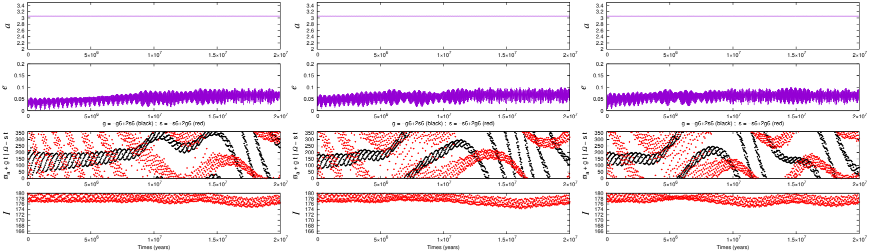

Figure 2: Respectively, evolution of the orbital elements using the initial location $a = 3.0615$ au with $\omega = 0°$, $a = 3.059$ au with $\omega = 90°$ and $a = 3.0597$ with $\omega = 180°$.

Region around $a = 1.28$ au

The relevant frequencies in this region are $g = g5 - g6 + g7$ and $s = s7 - g6 + g7$. Using the same method described above, we get the temporal evolution of the secular angles related to this frequencies. The values of the analyzed frequencies are described in Table 1.

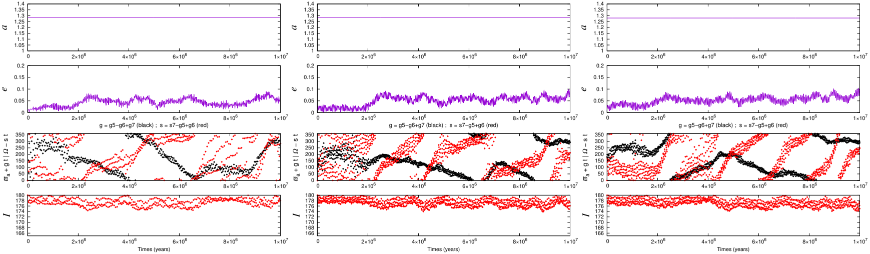

Figure 3: Respectively, evolution of the orbital elements using the initial location $a = 1.28536$ au with $\omega = 0°, 90°$ and $a = 1.28$ au with $\omega = 180°$.

Conclusion

In this work, we obtain an expansion of the second order perturbation function on the eccentricities and inclination for the retrograde case. From this result we developed a linear theory for secular resonance, that is, valid for small eccentricities and inclinations near 180 degrees.

A new analytical approach to the expansion of the disturbing function to the retrograde case allowed us to obtain the locations of secular resonances.

The numerical simulations using initial conditions around the calculated semi-major axis values confirmed the libration of the secular angles, which are predicted by the theoretical results.

References

[1] C.D. Murray S.D. Dermott (1999) Solar System Dynamics, Cambridge University Press

[2] M Carpino, A Milani, AM Nobili (1987) Long-term numerical integrations and synthetic theories for the motion of the outer planets, Astronomy and Astrophysics

[3] J.E.Chambers (1999) A Hybrid Symplectic Integrator that Permits Close Encounters between Massive Bodies